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SIMULTANEOUS LINEAR EXTENSION OF CONTINUOUS FUNCTIONS *

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A topological space X is said to have property D_c^*, where $c \geq 1$ is a real number, if for each closed subspace F of X there is a linear function $\Phi : C^*(F) \to C^*(X)$ such that (1) $\Phi(f)$ extends f, and (2) $\|\Phi\| \leq c$, i.e., $\|\Phi(f)\| \leq c\|f\|$ for $f \in C^*(F)$. We give a necessary condition for a space to have property D_c^* and then obtain the following results.

There exists a semimetrizable cosmic (hence hereditarily Lindelöf) space which does not have any property D_c^*.

For each odd integer $m > 1$ there exists a space which has property D_m^* but not property D_c^* for any $c < m$.

A space having property D_c^* for some $c < 3$ is hereditarily collectionwise normal. There is a compact Hausdorff space with property D_3^* which is not hereditarily normal.

βN does not have any property D_c^*.

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1. Introduction

Denote the vector space of all continuous real-valued functions on a space Y by $C(Y)$, and denote the vector space of all bounded continuous real-valued function on Y by $C^*(Y)$. The latter vector space is equipped with the sup-norm, and the norm of $f \in C^*(Y)$ will be denoted by $\|f\|$.

In [9], Dugundji proved the following theorem, which now bears his name.[1]

* This paper constitutes a part of the author's doctoral thesis.

[1] In fact, Dugundji and Borges proved their theorems for functions having values in a locally convex topological vector space.

Dugundji Extension Theorem. *Let F be a (nonempty) closed subspace of a metrizable space X. Then there is a linear function $\Phi : C(F) \to C(X)$ satisfying, for each $f \in C(F)$,*

(a) $\Phi(f) \upharpoonright F = f$, *i.e., Φ is an* extender.

(b) *the range of $\Phi(f)$ is a subset of the convex hull (in* R*) of the range of f.*

In [4], Borges generalized this result by showing that it is enough to assume the space X is stratifiable.[2] In this paper we study a family of properties related to the conclusion of the Dugundji Extension Theorem which can be used to answer some questions of Michael and Borges. These properties were introduced independently by Heath and Lutzer [16], [17].

A space X is said to have the property D_c^*, where $c \geqslant 1$ is a real number, if for each closed subspace F of X, there is a linear function $\Phi : C^*(F) \to C^*(X)$ satisfying, for each $f \in C^*(F)$,

(a) $\Phi(f) \upharpoonright F = f$

(b) the operator norm $\|\Phi\|$ of Φ does not exceed c, i.e.,

$$\|\Phi(f)\| \leqslant c\,\|f\| \quad \text{for each } f \in C^*(F)\,.$$

Thus the Dugundji Extension Theorem and Borges' generalization thereof imply that every metrizable space, and every stratifiable space, has property D_1^*. Recently Heath and Lutzer proved that every subspace of any linearly orderable space has property D_1^* [15].

In the 1950s Michael asked whether every perfectly space satisfies the Dugundji Extension Theorem. In his 1966 paper [4], Borges asked whether the members of the more restrictive class of perfectly paracompact spaces must satisfy this theorem. After Heath proved, in 1969, that every stratifiable space is a σ-space (see [14]), Michael specialized the question even further, asking whether paracompact σ-spaces (which have many properties in common with stratifiable spaces) satisfy the Dugundji Extension Theorem. Our example H_∞, presented in Section 4, settles all of these questions negatively: there exist Lindelöf, first countable σ-spaces which do not have property D_c^* for any c; certainly such spaces cannot satisfy the conclusion of the Dugundji Extension Theorem.

[2] It has been observed by Heath and Lutzer that for each closed subspace F of a normal space X there exists a linear extender $C(F) \to C(X)$, [15] (apply the Tietze–Urysohn extension theorem to the functions of a Hamel base for $C(F)$). Analogously one can find a linear extender $C^*(F) \to C^*(X)$.

We shall give a necessary condition for a space to have property D_c^*. In order to explain our condition, it is convenient to denote the topology on a space Y by τY (occasionally $\tau(Y)$ for clarity). Then a space X is collectionwise normal if (and only if) for each discrete family $\mathscr{A}$ in X, consisting of closed sets, there is a function $\kappa : \mathscr{A} \to \tau X$ such that:

(1) $(\bigcup \mathscr{A}) \cap \kappa(A) = A$, for $A \in \mathscr{A}$,

(2) $\kappa(A) \cap \kappa(B) = \emptyset$ whenever $A \neq B$, for $A, B \in \mathscr{A}$.

Observing that each $A \in \mathscr{A}$ is open in the closed subspace $\bigcup \mathscr{A}$, we see that a sufficient condition for a space X to be collectionwise normal is the following:

For each closed subspace F of X there is a function κ: $\tau F \to \tau X$ such that:

(1) $F \cap \kappa(V) = V$, for $V \in \tau F$,

(2) $\kappa(V) \cap \kappa(W) = \emptyset$ whenever $V \cap W = \emptyset$, for $V, W \in \tau F$.

(κ "extends" open sets of F to open sets of X in such a way that extensions of disjoint open sets of F are again disjoint). Spaces satisfying this condition will be called K_1*-spaces*. It turns out that in a K_1-space X there is for *each* subspace F a function κ: $\tau F \to \tau X$ satisfying (1) and (2) above; see 2.1. Therefore a subspace of a K_1-space is a K_1-space; see 2.2(1). Hence K_1-spaces are hereditarily collectionwise normal. We will show that a space with property D_c^* for some $c < 3$ is a K_1-space. On the other hand, there is a compact space with property D_3^* which is not even hereditarily normal.

The concept of K_1-space was motivated by a theorem of Kuratowski, stating that for each subspace F or a metrizable space X there is a function $\kappa : \tau F \to \tau X$ such that:

(1) $F \cap \kappa(V) = V$, for $V \in \tau F$,

(2) $\kappa(V) \cap \kappa(W) = \kappa(V \cap W)$, for $V, W \in \tau F$,

(3) $\kappa(\emptyset) = \emptyset$

[18, p. 122, § 15, XIII]. We will study spaces with this property, K_1-spaces and relations with extension properties in [7].

Generalizing the concept of K_1-space, we say that X is a K_n*-space* if, for each closed subspace F of X (equivalently, for each subspace F of X; see 2.1), there exists a function $\kappa : \tau F \to \tau X$ such that

(1) $F \cap \kappa(V) = V$, for $V \in \tau F$,

(2) $\kappa(V_0) \cap ... \cap \kappa(V_n) = \emptyset$ whenever $V_i \cap V_j = \emptyset$, for $0 \leq i < j \leq n$, $V_0, ..., V_n \in \tau F$.

The generalization to K_n-spaces was motivated by a theorem of Heath and Lutzer [16], who showed that any space with property D_c^* has the property (depending on n) described in 3.3 with $n > c - 1$.

We shall see that a necessary condition for a space X to have property D_c^* is that X be a K_n-space, where n is the smallest integer $> \frac{1}{2}(c-1)$. This is best possible in the following sense: for each positive integer n there exists a space P_n with property D_{2n+1}^* which is not a K_n-space. The spaces P_n also show that, for each odd integer $m > 1$, there is a space having property D_m^*, but not property D_c^* for any $c < m$. Up to now, only spaces with property D_1^* and spaces not having any property D_c^* were known.

Our condition is not sufficient: a space having some property D_c^* is normal, but there exist nonnormal K_2-spaces; see 2.3. This leaves open the possibility that K_1-spaces have property D_1^*. However, this is highly improbable, for, by a recent result of Benyamini [2], there is for each $\lambda > 1$ a compact space X_λ having a closed subspace F_λ such that

$$\lambda = \min\{\|\Phi\| \mid \Phi : C^*(F_\lambda) \to C^*(X_\lambda) \text{ is a linear extender}\}$$

(Earlier such pairs $\langle X_\lambda, F_\lambda\rangle$ were constructed by Corson and Lindenstrauss [5] for odd integers $\lambda > 1$.) Then there is a K_1-function $\kappa : \tau F_2 \to \tau X_2$, but not a linear extender $C^*(F_2) \to C^*(X_2)$ with operator norm 1. We do not know whether there exists for each real number $\lambda > 1$ a space having property D_λ^*, but not property D_c^* for any $c < \lambda$.

It follows from our result that a space with property D_c^* for some $c < 3$ is hereditarily collectionwise normal (since K_1-spaces are hereditarily collectionwise normal). The converse of this statement is far from being true. There is a first countable cosmic (hence hereditarily Lindelöf) space H_∞ which does not have property D_c^* for any c.[2] H_∞ is the topological sum of a family $\{H_n \mid n \in \mathbb{N}\}$, where each H_n is a first countable cosmic space which is a K_{n+1}-space but not a K_n-space. The existence of such a family is of interest for yet another reason: it shows that the property of being a K_n-space is independent of such properties as being hereditarily Lindelöf.

The structure of the invidual spaces H_n is not completely understood; for example, while we know that H_1 does not have property D_c^* for any $c < 3$, we do not know whether H_1 has property D_c^* for some $c \geqslant 3$ (we believe it has not).

We assume that all spaces are at least T_1. Then a space having some property D_c^* is completely regular. This will be used without explicit mention. $\mathbb{N}$ is the set of *positive* integers, $\mathbb{Z}$ is the set of all integers, and $\mathbb{R}$ is the space of real numbers. A *cosmic* space is a regular space with a countable network. Here a family $\mathscr{A}$ is called a *network* for a space X if, whenever U is a neighborhood of a point $p \in X$, $p \in A \subset U$ for some $A \in \mathscr{A}$.

2. K_n-spaces

Throughout this section, n is an arbitrary nonnegative integer. Let F be a subspace of a space X. We call a function $\kappa : \tau F \to \tau X$ a K_n-*function* if:

(1) $F \cap \kappa(V) = V$, for $V \in \tau F$;

(2) if $n = 0$, then $\kappa(\emptyset) = \emptyset$ and $\kappa(V) \cap \kappa(W) = \kappa(V \cap W)$, for $V, W \in \tau F$, if $n > 0$, then $\kappa(V_0) \cap ... \cap \kappa(V_n) = \emptyset$ whenever $V_i \cap V_j = \emptyset$, for $0 \leqslant i < j \leqslant n$, $V_0, ..., V_n \in \kappa F$.

Taking $V_0 = ... = V_n = \emptyset$ in the above definition, we see that $\kappa(\emptyset) = \emptyset$, also if $n > 0$. We call a space X a K_n-*space* if there is for each closed subspace F of X a K_n-function $\kappa : \tau F \to \tau X$. Clearly a K_n-space is a K_{n+1}-space, for $n \geqslant 0$. The converse is not true for $n \geqslant 1$ (see Sections 4 and 5). We do not know whether K_1-spaces are K_0-spaces. The concept K_0-function is used in an essential way in 4.5.

2.1. Lemma. *For a space X the following conditions are equivalent*:

(a) *for each subspace F of X there is a K_n-function $\kappa : \tau F \to \tau X$*;

(b) *for each closed subspace F of X there is a K_n-function $\kappa : \tau F \to \tau X$ (i.e., X is a K_n-space)*.

Proof. It suffices to prove that (b) implies (a). If F is a subspace of X let $\lambda : \tau F^- \to \tau X$ be a K_n-function, and define $\kappa : \tau F \to \tau F^-$ by $\kappa(V) = F^- - (F - V)^-$. Then κ is a K_0-function. It follows that $\lambda \circ \kappa : \tau F \to \tau X$ is a K_n-function. □

2.2. Corollary. (1) *A subspace of K_n-space is a K_n-space.*

(2) *A K_1-space is hereditarily collectionwise normal.*

Proof. (1) If $F \subset Y \subset X$ and $\kappa : \tau F \to \tau X$ is a K_n-function, then $\lambda : \tau F \to \tau Y$ defined by $\lambda(V) = Y \cap \kappa(V)$ is also a K_n-function.

(2) Since K_1-spaces are collectionwise normal, this follows from (1). Alternatively, if $\mathscr{A}$ is a discrete collection of closed subsets in some subspace of X, consider a K_1-function $\kappa : \tau(\bigcup\mathscr{A}) \to \tau X$. □

2.3. In contrast to Corollary 2.2(2), a K_2-space need not even be Hausdorff (remember that all spaces are supposed to be T_1):

Let $S = \mathbf{N} \cup \{p, q\}$, where p and q are two points not in $\mathbf{N}$. Each point of $\mathbf{N}$ is isolated, and a neighborhood of p or q contains all but finitely many points of S (S is a sequence converging to two points). S is a K_2-space but it is not Hausdorff.

The referee kindly supplied the following example of a normal K_2-space which is not collectionwise normal. Let X be the set of all points in the upper half-plane, including the x-axis. Points above the x-axis are isolated, and points $\langle x, 0\rangle$ on the x-axis have basic neighborhoods of the form

$$\{\langle x, 0\rangle\} \cup \{\langle r, s\rangle \mid 0 < s < \epsilon, \text{ the line from } \langle x, 0\rangle \text{ to } \langle r, s\rangle \text{ has slope } \pm 1\}\ .$$

X is described by Heath [11, p. 175, Example 1] and is a nonnormal K_2-space. Assuming Martin's axiom + $2^{\aleph_0} > \aleph_1$, there is a normal subspace which is not collectionwise normal. There is an uncountable subset G of the x-axis such that each subset of G is an F_σ in G, in the relative plane topology on G [22, p. 74, 1.19] or [21, p. 119]. Then $G \cup \{\langle x, y\rangle \mid y > 0\}$ is normal, but not collectionwise normal. This is essentially [3, p. 182, Example E]. This subspace is also a K_2-space.

3. Simultaneous linear extenders

3.1. Main Theorem. *A space with property D_c^* is a K_n-space, where n is the smallest integer $> \frac{1}{2}(c-1)$.*

Proof. Let F be a closed subspace of a space X having property D_c^*, and assume $c < 2n + 1 - \epsilon$, where $\epsilon > 0$. Let Φ be a linear extender from $C^*(F)$ to $C^*(X)$ with norm not exceeding $2n + 1 - \epsilon$. We have to construct a K_n-function $\kappa : \tau F \to \tau X$.

Of course, we have in general the inequality

$$|\Phi(f)(x)| \leqslant (2n + 1 - \epsilon) \cdot \|f\|, \quad \text{for } x \in X, f \in C^*(X).$$

But if we restrict ourselves to *nonnegative* functions, a better inequality is possible for points in a suitable *fixed* neighborhood of F. We formulate this as a lemma. In 5.5 we explain the origin of this idea.

3.2. Lemma. *Let $\Phi : C^*(F) \to C^*(X)$ be a linear extender with norm not exceeding $a - \epsilon$, where $\epsilon > 0$ (and $a - \epsilon \geqslant 1$). Then there is an open neighborhood W of F such that $|\Phi(f)(x)| < (\frac{1}{2}(a+1) - \frac{1}{4}\epsilon) \cdot \|f\|$, for $x \in W$ and every nonnegative $f \in C^*(F)$.*

Proof. Let $e : F \to R$ be the function defined by $e(x) = 1$ for $x \in F$. Let $W = \Phi(e)^{-1}[(1 - \frac{1}{2}\epsilon, 1 + \frac{1}{2}\epsilon)]$. [We cannot be sure that $\Phi(e)(x) = 1$ for all $x \in X$!.] Since Φ is linear, $\Phi(\alpha \cdot e) = \alpha \cdot \Phi(e)$ for each real number α. Observing that $\|f - \frac{1}{2}\|f\| \cdot e\| = \frac{1}{2}\|f\|$ for nonnegative bounded functions on F, we see that for $x \in W$ and for nonnegative $f \in C^*(F)$ the following holds:

$$
\begin{aligned}
|\Phi(f)(x)| &= |\Phi(f - \tfrac{1}{2}\|f\| \cdot e)(x) + \Phi(\tfrac{1}{2}\|f\| \cdot e)(x)| \\
&\leqslant |\Phi(f - \tfrac{1}{2}\|f\| \cdot e)(x)| + \tfrac{1}{2}\|f\| \cdot |\Phi(e)(x)| \\
&< (a - \epsilon) \cdot \tfrac{1}{2}\|f\| + \tfrac{1}{2}\|f\| \cdot (1 + \tfrac{1}{2}\epsilon) \\
&= (\tfrac{1}{2}(a + 1) - \tfrac{1}{4}\epsilon) \cdot \|f\|. \square
\end{aligned}
$$

Proof of the theorem (continued). Denote the set of all continuous functions from F to the unit interval $[0,1]$ by C^{**}. Let W be as in the lemma with $a = 2n + 1$. Define a function $\kappa : \tau F \to \tau X$ by

$$
\kappa(V) = W \cap [\bigcup\{\Phi(f)^{-1}[(1 - \epsilon/4(n+1), \infty)] \mid f \in C^{**},\ f[F - V] \subset \{0\}\}] .
$$

[Observe that "$\subset$", instead of "$=$", ensures that $\kappa(F) \supset F$.] Since Φ is an extender, $F \cap \kappa(V) = V$ for $V \in \tau F$.

Let $V_0, ..., V_n \in \tau F$ satisfy $V_i \cap V_j = \emptyset$ for $0 \leqslant i < j \leqslant n$. Let $f_0, ..., f_n$ be arbitrary functions of C^{**} satisfying $f_i[F - V_i] \subset \{0\}$ for $0 \leqslant i \leqslant n$. Then $f = f_0 + ... + f_n$ is nonnegative, and $\|f\| = 1$. Hence for $x \in W$

$$
\Phi(f_0)(x) + ... + \Phi(f_n)(x) \leqslant n + 1 - \tfrac{1}{4}\epsilon
$$

since Φ is linear. Therefore

$$
W \cap \bigcap_{i=0}^{n} \Phi(f_i)^{-1}[(1 - \epsilon/4(n+1), \infty)] = \emptyset .
$$

Hence $\kappa(V_0) \cap ... \cap \kappa(V_n) = \emptyset$. This proves that κ is a K_n-function. $\square$

In Section 5 we show that this theorem is best possible in the following sense: For each $n \in \mathbf{N}$ there exists a space P_n with property D^*_{2n+1} which is not a K_n-space.

3.3. Corollary. *Let X have property D^*_c, let n be the smallest integer $> \frac{1}{2}(c-1)$. Then for every discrete collection $\mathcal{A}$ of closed sets in X there is an open collection $\{\kappa(A) \mid A \in \mathcal{A}\}$ such that:*

(1) $A \subset \kappa(A)$, for $A \in \mathcal{A}$;

(2) $\kappa(A) \cap B = \emptyset$, for $A, B \in \mathcal{A}$, $A \neq B$;

(3) any point of X belongs to at most n sets $\kappa(A)$.

This was proved by Heath and Lutzer with $c-1$ instead of $\frac{1}{2}(c-1)$ [16] (announced for $c < 2$ in [17]). We do not know whether this corollary is best possible. We do not even know whether there exists a space with some property D^*_c which is not collectionwise normal. However, we do have the following result, by 3.1 and 2.2(2).

3.4. Corollary. *A space with property D^*_c for some $c < 3$ is hereditarily collectionwise normal.*

There is a compact Hausdorff space with property D^*_3 which is not hereditarily normal (see Section 5). Thus this corollary is best possible. The converse of the corollary is far from being true. In Section 4 we construct a first countable cosmic space which does not have property D^*_c for any c.

3.5. Corollary. *$\beta\mathbf{N}$ does not have property D^*_c for any c.*

Proof. $\beta\mathbf{N} - \mathbf{N}$ contains a family of cardinality $2^{\aleph_0}$ consisting of pairwise disjoint open sets, [10, p. 133, Example 2]. It follows that there can be no K_n-function $\kappa : \tau(\beta\mathbf{N} - \mathbf{N}) \to \tau(\beta\mathbf{N})$ for any n, since the countable set $\mathbf{N}$ is dense. □

According to [16], this has been proved by Banilower [1]. For $c = 1$ the fact was known earlier (see e.g. [11]).

We next show that the condition that for each closed subspace of a space X there is an additive extender mapping nonnegative functions to nonnegative functions is remarkably strong, and implies that X has property D^*_1, from which it follows that X is a K_1-space, by Theorem 3.1.

If Y is a space, and if S is any nonempty subset of $\mathbf{R}$, we denote by $C(Y, S)$ and $C^*(Y, S)$ the set of all continuous, respectively bounded continuous, functions $Y \to S$.

We will call an extender Φ a *(closed) convex hull extender* if the range of $\Phi(f)$ is contained in the (closed) convex hull of the range of f, for each f in the domain of Φ.

3.6. Theorem. *Let F be a closed subspace of a normal space X.*

(a) *There is a linear convex hull extender $\Phi : C(F) \to C(X)$ iff there is an additive extender $\Psi : C(F, (0,\infty)) \to C(X, (0,\infty))$.*

(b) *There is a linear convex hull extender $\Phi : C^*(F) \to C^*(X)$ if there is an additive extender $\Psi : C^*(F, (0,\infty)) \to C^*(X, (0,\infty))$.*

(c) *There is a linear closed convex hull extender $\Phi : C(F) \to C(X)$ iff there is an additive extender $\Psi : C(F, [0,\infty)) \to C(X, [0,\infty))$.*

(d) *There is a linear closed convex hull extender $\Phi : C^*(F) \to C^*(X)$ iff there is an additive extender $\Psi : C^*(F, [0,\infty)) \to C^*(X, [0,\infty))$.*

Pełczynski has proved (d) under the additional condition that $\Psi(\alpha f) = \alpha \Psi(f)$ for $\alpha > 0$ [20, p. 19, Proposition 2.11].

Let M be the Michael line, let $\mathbf{Q}$ be the (closed) subspace of rationals of M. Then there does not exist a linear closed convex hull extender $C(\mathbf{Q}) \to C(M)$ (hence not a linear convex hull extender $C(\mathbf{Q}) \to C(M)$) [15] and there does not exist a linear convex hull extender $C^*(\mathbf{Q}) \to C^*(M)$ [7]. However, for each closed subspace F of M there is a linear closed convex hull extender $C^*(F) \to C^*(M)$ [15] and M can be embedded in a space satisfying the Dugundji Extension Theorem [7]. It is not known whether the property that for each closed F in X there is a linear closed convex hull extender $C^*(F) \to C^*(X)$ is equivalent to property D_1^*.

Proof of Theorem 3.6. It suffices to prove that the conditions are sufficient. Let 1_F and 1_X have the obvious meaning. We first show that we may assume that $\Psi(1_F) = 1_X$ in all four cases, and then prove (a). The remaining parts can be proved in a similar way.

First step. Let Ψ be an extender given in one of the four cases. Let $W = \Psi(1_F)^{-1}[(\frac{1}{2}, \infty)]$ and $w : X \to [0, 1]$ be a continuous function satisfying

$$F \subset w^{-1}[\{0\}], \qquad X - W \subset w^{-1}[\{1\}] .$$

Pick any point $x_0 \in F$ and define Ψ_1 and Ψ_2 by

$$\Psi_1(f) = \Psi(f) + f(x_0) \cdot w,$$
$$\Psi_2(f) = \Psi_1(f)/\Psi_1(1_F).$$

Then Ψ_1 and Ψ_2 are additive extenders with the same domain as Ψ, and $\Psi_2(1_F) = 1_X$. [Observe that $\Psi_1(1_F)(x) > \frac{1}{2}$ for $x \in X$.] Hence we may assume that $\Psi(1_F) = 1_X$.

Second step. Since $\Psi(1_F) = 1_X$, and Ψ is additive, we can define an extender

$$\Psi' : C(F, [0,\infty)) \to C(X, [0,\infty)),$$

satisfying $\Psi' \restriction C(F, (0,\infty)) = \Psi$, by

$$\Psi'(f) = \Psi(1_F + f) - 1_X.$$

Then:

(a) $\Psi'[C(F, (0,\infty))] \subset C(X, (0,\infty))$.

Next define an extender $\Phi : C(F) \to C(X)$ by

$$\Phi(f) = \Psi'(f^+) - \Psi'(-f^-)$$

where, as usual, $f^+(x) = \max\{0, f(x)\}$ and $f^-(x) = \min\{0, f(x)\}$. Then Φ is additive. [Ψ' is additive, hence $\Psi'(a) - \Psi'(b) = \Psi'(c) - \Psi'(d)$ if $a - b = c - d$, for $a, b, c, d \in C(F, [0,\infty))$. Additivity of Φ easily follows.] Now $\Phi \restriction C(F, [0,\infty)) = \Psi'$; so $\Phi[C(F, [0,\infty))] \subset C(X, [0,\infty))$. Using (a) we infer:

(b) if $f \leqslant g$, then $\Phi(f) \leqslant \Phi(g)$; if $f < g$, then $\Phi(f) < \Phi(g)$;

since Φ is additive. Since $\Phi(\alpha f) = \alpha\, \Phi(f)$ for rational α, (b) implies that this equality holds for all real numbers α. Therefore Φ is linear, and

(c) $\Phi(\alpha\, 1_F) = \alpha\, 1_X$ for all $\alpha \in \mathbb{R}$.

By (b) and (c), Φ is a convex hull extender. □

4. A cosmic space without any property D_c^*

In this section we construct a first countable cosmic space H_∞ which does not have property D_c^* for any c. Observe that H_∞ is semimetrizable, for any cosmic space is semistratifiable, and by a theorem of Creede a space is semimetrizable if and only if it is first countable and semistratifiable [6].[3]

In order to construct H_∞, we construct for each positive integer n a first countable cosmic K_{n+1}-space H_n which is not a K_n-space Then by 2.2(1) the topological sum H_∞ of the family $\{H_n \mid n \in \mathbf{N}\}$ is not a K_n-space for any n and hence H_∞ does not have property D_c^* for any c, by Theorem 3.1.

The construction of the spaces H_n is by induction on n. Throughout this section, for $a, b \in \mathbf{R} \cup \{-\infty, +\infty\}$ we denote by $I(a, b)$ the subset

$$\{x \in \mathbf{R} \mid x \text{ irrational and } a < x < b\} \times \{0\}$$

of $\mathbf{R}^2$, and by d the Euclidean metric in $\mathbf{R}^2$. The coordinates of a point $x \in \mathbf{R}^2$ will be denoted by x_1 and x_2.

4.1. Construction of H_1.[4] The underlying set of H_1 is the subset $A \cup B \cup C$ of $\mathbf{R}^2$, where

$$\begin{aligned}
A &= I(-\infty, +\infty),\\
B &= \{\langle p/q,\ 1/q\rangle \mid p \in \mathbf{Z},\ q \in \mathbf{N}\},\\
C &= \{x \in \mathbf{R}^2 \mid \text{both } x_1 \text{ and } x_2 \text{ are rational}, x_2 > 0\} - B.
\end{aligned}$$

The topology of H_1 is determined as follows. Basic (open) neighborhoods in H_1 of a point $x \in H_1$ are

$$U_1(x, m) = \begin{cases} \{y \in H_1 \mid y_2 \leq |x_1 - y_1| < 1/m\} & \text{if } x \in A,\\ \{x\} \cup \{y \in C \mid d(x, y) < 1/m\} & \text{if } x \in B,\\ \{x\} & \text{if } x \in C,\end{cases}$$

for $m \in \mathbf{N}$.

Regularity is easily checked because for any $a \in A$ the boundary of $U_1(a, m)$ in H_1 is the set $\{\langle a_1 - 1/m, 0\rangle, \langle a_1 + 1/m, 0\rangle\}$ (in fact, ind $H_1 = 0$). H_1 is the union of the separable metrizable subspace A, B and C, and hence has a countable network. Therefore H_1 is cosmic.

Observe that C, the set of isolated points in H_1, is dense in H_1.

4.2. Proof that H_1 is not a K_1-space. Let $F_1 = A \cup B$, and let $\kappa : \tau F_1 \to \tau H_1$ be any function such that $F_1 \cap \kappa(V) = V$ for $V \in \tau F_1$. Since the subspace A is a Baire space, for some $k \in \mathbf{N}$ the set

[3] This reference is due to the referee.

[4] This space is a minor modification of an example by Heath [13]. It is a kind of a butterfly space. In the same way as in [8, Example 2.4], one can show that H_1 is a quotient of a first countable separable stratifiable space.

$$L_k = \{x \in A \mid U_1(x, k) \subset \kappa(F_1 \cap U_1(x, 1))\}$$

is not nowhere dense in A. Hence there exist $s, t \in \mathbf{R}$ with $s < t$ such that $L_k \cap I(s, t)$ is dense in $I(s, t)$. Select $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ such that

$$1/q < 1/k, \qquad s < (p-1)/q, \qquad p/q < t$$

and let z be the point $\langle p/q, 1/q\rangle$. Since $\{z\} \in \tau F_1$, there is an $m \geqslant q$ such that $U_1(z, m) \subset \kappa(\{z\})$. Find a $y \in L_k \cap I((p-1)/q, (p-1)/q + 1/m)$. Then $U_1(z, m) \cap U_1(y, k) \neq \emptyset$ and thus $\kappa(\{z\}) \cap \kappa(F_1 \cap U_1(y, 1)) \neq \emptyset$ although evidently $\{z\} \cap (F_1 \cap U_1(y, 1)) = \emptyset$. It follows that κ is not a K_1-function, hence H_1 is not a K_1-space.

4.3. Construction of H_{n+1} for $n \in \mathbf{N}$. Assume that we have constructed a first countable cosmic space H_n which is not a K_n-space and in which the (necessarily countable) set I_n of isolated points is dense. Let $\{U_n(x, m) \mid m \in \mathbf{N}\}$ be a neighborhood base in H_n, for $x \in H_n$.

The underlying set of H_{n+1} will be $A \cup (B \times H_n) \cup C$. For convenience we denote $B \times H_n$ by B_{n+1}.

There is a pairwise disjoint collection $\{T(b, x, m) \mid b \in B, x \in I_n, m \in \mathbf{N}\}$ of subsets of C such that each $T(b, x, m)$ is of the form

$$\{y \in C \mid \alpha < d(b, y) < \beta\}$$

where α and β are real numbers, depending on b, x and m, such that $0 < \alpha < \beta < 1/m$. The topology of H_{n+1} is determined as follows. Basic (open) neighborhoods in H_{n+1} of a point $x \in H_{n+1}$ are

$$U_{n+1}(x, m) = \begin{cases} (U_1(x, m) - B) \cup \bigcup\{\{b\} \times H_n \mid b \in B \cap U_1(x, m)\} \\ \qquad\qquad \text{if } x \in A, \\ (\{b\} \times U_n(w, m)) \cup \bigcup\{T(b, y, k) \mid y \in I_n \cap U_n(w, m), k \geqslant m\} \\ \qquad\qquad \text{if } x = \langle b, w\rangle \in B_{n+1}, \\ \{x\} \qquad\quad \text{if } x \in C, \end{cases}$$

for $m \in \mathbf{N}$.

The straightforward proof that the families $\{U_{n+1}(x, m) \mid m \in \mathbf{N}\}$ are indeed neighborhood systems and that they generate a regular topology on H_{n+1} is omitted. H_{n+1} is the union of countably many cosmic subspaces, hence has a countable network. It follows that H_{n+1} is cosmic.

Observe that the set $I_{n+1} = C$ of isolated points in H_{n+1} is dense.

4.4. Proof that H_{n+1} is not a K_{n+1}-space, for $n \in \mathbb{N}$. Since H_n is not a K_n-space, there is an $F_n \subset H_n$ such that there is no K_n-function $\kappa : \tau F_n \to \tau H_n$. Let $F_{n+1} = A \cup (B \times F_n)$, and let $\kappa : \tau F_{n+1} \to \tau F_{n+1}$ be any function such that $F_{n+1} \cap \kappa(V) = V$, for $V \in F_{n+1}$. In the same way as in 4.2 we can find a $k \in \mathbb{N}$ and $z = \langle p/q, 1/q \rangle \in B$ such that if $m \geq q$ and $t \in I_n$, then there is a $y \in A$ satisfying:

(1) $U_{n+1}(y, k) \subset \kappa(F_{n+1} \cap U_{n+1}(y, 1))$,

(2) $U_{n+1}(y, k) \cap (\{z\} \times H_n) = \emptyset$,

(3) $T(z, t, m) \cap U_{n+1}(y, k) \neq \emptyset$.

[Note that $m \geq q$ (hence $\operatorname{diam}(T(b, t, m)) \leq z_2$) is essential to achieve (3).]

Since $\{z\} \times F_n$ is open in F_{n+1}, we can define a function $\lambda : \tau(\{z\} \times F_n) \to \tau(\{z\} \times H_n)$ by $\lambda(V) = (\{z\} \times F_n) \cap \kappa(V)$. Then obviously $(\{z\} \times F_n) \cap \lambda(V) = V$ for $V \in \tau(\{z\} \times F_n)$.

Now the map $h : H_n \to \{z\} \times H_n$ defined by $h(x) = \langle z, x \rangle$ is a homeomorphism. Hence λ is not a K_n-function. Therefore there are $V_0, \ldots, V_n \in \tau(\{z\} \times F_n)$ such that:

(4) $V_i \cap V_j = \emptyset$, for $0 \leq i < j \leq n$, and yet $\bigcap_{i=0}^{n} \lambda(V_i) \neq \emptyset$.

Then there is a $t \in I_n$ with $\langle z, t \rangle \in \bigcap_{i=0}^{n} \lambda(V_i)$, hence $\langle z, t \rangle \in \bigcap_{i=0}^{n} \kappa(V_i)$. Therefore there is an $m \in \mathbb{N}$ with $m \geq q$ such that $T(z, t, m) \subset \bigcap_{i=0}^{n} \kappa(V_i)$. Choose a $y \in A$ satisfying (1), (2) and (3). Let $V_{n+1} = F_{n+1} \cap U_{n+1}(y, 1)$. Then $\bigcap_{i=0}^{n+1} \kappa(V_i) \supset T(z, t, m) \cap U_{n+1}(y, k) \neq \emptyset$ although $V_i \cap V_j = \emptyset$ for $0 \leq i < j \leq n+1$ by (2) and (4) (since $(\{z\} \times H_n) \cap V_{n+1} = \emptyset$, and thus $V_i \cap V_{n+1} = \emptyset$ for $0 \leq i \leq n$). It follows that κ is not a K_{n+1}-function, hence H_{n+1} is not a K_{n+1}-space.

4.5. Proof that H_n is a K_{n+1}-space for $n \in \mathbb{N}$. For $n > 1$ we denoted $H_n - (A \cup C) = B \times H_{n-1}$ by B_n; in addition we will denote $H_1 - (A \cup C) = B$ by B_1.

The proof that H_n is a K_{n+1}-space is by induction on n. Clearly B_1, being discrete, is a K_1-space. If we assume that H_n is a K_{n+1}-space, then so is B_{n+1}, because B_{n+1} is the topological sum of copies of H_n. Hence in order to prove both the first and the inductive step it suffices to prove:

(*) *If B_n is a K_n-space, then H_n is a K_{n+1}-space.*

Let $n \in \mathbb{N}$ be fixed, and let G be a closed subspace of H_n. We can define a K_1-function $\lambda : \tau A \to \tau H_n$ by

$$\lambda(V) = \bigcup \{U_n(x, m) \mid x \in A,\ A \cap U_n(x, m) \subset V,\ m \in \mathbb{N}\}.$$

As noted in the introduction, Kuratowski proved that metrizable spaces are K_0-spaces, [18, p. 122, § 15, XIII], hence there is a K_0-function $\varphi : \tau(A \cap G) \to \tau A$. Define $\gamma : \tau G \to \tau H_n$ by

$$\gamma(V) = [(\lambda \circ \varphi)(A \cap V)] - (G - V).$$

Then

(5) $\gamma(V) \supset A \cap V$ and $G \cap \gamma(V) \subset V$ for $V \in \tau G$,

(6) $\gamma(V) \cap \gamma(W) = \emptyset$ if $V \cap W = \emptyset$, for $V, W \in \tau G$.

[Observe that subtraction of $G - V$ is essential to achieve the second part of (5).] We can define a K_0-function $\mu : \tau B_n \to \tau H_n$ by

$$\mu(V) = \begin{cases} \bigcup\{U_1(\langle p/q, 1/q\rangle, 2q) \mid \langle p/q, 1/q\rangle \in V\} & \text{if } n = 1 \\ V \cup \bigcup\{T(b,x,m) \mid x \in I_{n-1}, \langle b,x\rangle \in V, m \in \mathbb{N}\} & \text{if } n > 1. \end{cases}$$

Since B_n is a K_n-space, there is a K_n-function $\psi : \tau(B_n \cap G) \to \tau B_n$. Define a function $\delta : \tau G \to \tau H_n$ by

$$\delta(V) = [(\mu \circ \psi)(B_n \cap V)] - (G - V).$$

Then

(7) $\delta(V) \supset B_n \cap V$ and $G \cap \delta(V) \subset V$ for $V \in \tau G$,

(8) $\delta(V_0) \cap \ldots \cap \delta(V_n) = \emptyset$ if $V_i \cap V_j = \emptyset$ for $0 \leqslant i < j \leqslant n$, for $V_0, \ldots, V_n \in \tau G$.

[Observe that if $V_i \cap V_j = \emptyset$ for $0 \leqslant i < j \leqslant n$, for $V_0, \ldots, V_n \in \tau G$, then

$$\bigcap_{i=0}^{n} \delta(V_i) \subset \bigcap_{i=0}^{n} \mu(\psi(B_n \cap V)) = \mu\left(\bigcap_{i=0}^{n} \psi(B_n \cap V_i)\right) = \mu(\emptyset) = \emptyset$$

since μ is a K_0-function.] Finally, we can define a function $\kappa : \tau G \to \tau H_n$ by

$$\kappa(V) = \gamma(V) \cup \delta(V) \cup (C \cap V).$$

Then, by (5) and (7):

(9) $G \cap \kappa(V) = V$ for $V \in \tau G$.

If $V, W \in \tau G$ are disjoint, then

$$\kappa(V) \cap \kappa(W) = [\gamma(V) \cup \delta(V)] \cap [\gamma(W) \cup \delta(W)]$$

by (9), and hence, by (6) and (8):

(10) $\kappa(V_0) \cap \ldots \cap \kappa(V_{n+1}) = \emptyset$ if $V_i \cap V_j = \emptyset$ for $0 \leqslant i < j \leqslant n+1$, $V_0, \ldots, V_{n+1} \in \tau G$.

It follows that κ is a K_{n+1}-function. Hence H_n is a K_{n+1}-space.

5. Spaces with property D_m^* for odd m.

We construct for every positive integer n a space P_n with property D_{2n+1}^* which is not a K_n-space, and hence does not have property D_c^* for any $c < 2n + 1$ by Theorem 3.1. In fact, P_n will have a property which is stronger than D_{2n+1}^* but weaker than each D_c^* with $c < 2n+1$, viz. property D_{2n+1}^{**}:

A space X is said to have property D_c^{**}, where $c > 1$ is a real number, if for each closed subspace F of X there is a linear extender $\Phi : C^*(F) \to C^*(X)$ such that $\|\Phi(f)\| < c \cdot \|f\|$ for $f \in C^*(F)$, provided $\|f\| \neq 0$ (of course, $\|\Phi(f)\| = 0$ if $\|f\| = 0$ since Φ is linear).

5.1. Construction of P_n. Let A_i be a discrete space of cardinality $\aleph_i$. The one-point compactification of A_i will be denoted by C_i, and the point at infinity will be called p_i, $i = 0, 1, \ldots$. Let n be a fixed positive integer.

Q_n is the "cube" $\Pi_{i=0}^n C_i$, and $I_n = \Pi_{i=0}^n A_i$ is its "interior". The coordinates of a point $x \in Q_n$ are denoted by $x_0, \ldots, x_n$. The "origin" is the point q defined by

$$q_j = p_j, \quad 0 \leqslant j \leqslant n.$$

For $x \in A_i$, the point $e(i, x) \in Q_n$ is defined by

$$e(i,x)_j = \begin{cases} x & \text{if } j = i\,, \\ p_j & \text{if } j \neq i\,. \end{cases}$$

The i^{th} "edge" is the set

$$E_i = \{e(i,x) \mid x \in A_i\}$$

(observe that $q \notin E_i$). The subspace

$$P_n = \{q\} \cup \bigcup_{i=0}^{n} E_i \cup I_n$$

of Q_n is the space we are looking for. Observe that $P_1 = Q_1$ and hence P_1 is compact; however, $P_n \neq Q_n$ if $n \neq 1$. It is left as an exercise to the reader to show that each P_n is paracompact. Throughout, we will denote by B_n the "boundary" of P_n, i.e., the closed subspace $P_n - I_n = \{q\} \cup \bigcup_{i=1}^{n} E_i$.

5.2. Proof that P_n is not a K_n-space. The "edges" E_i are pairwise disjoint open subsets of the "boundary" B_n. Let U_i be a neighborhood of E_i in P_n, for $0 \leqslant i \leqslant n$. In order to prove that there is no K_n-function $\kappa : \tau B_n \to \tau P_n$, it suffices to prove that $\bigcap_{i=0}^{n} U_i \neq \emptyset$. The proof is an extension of the proof for the case $n = 1$, which is well known.

If $x \in A_i$, then $e(i, x) \in E_i \subset U_i$; hence for $j \neq i$ there are finite subsets $F(i, x; j)$ of A_j such that for $y \in I_n$ the following holds:

(α) $y \in U_i$ if $y_i = x$ and $y_j \in A_j - F(i, x; j)$ for $j \neq i$.

Let $F_0 = \emptyset$, and define for $j \geqslant 1$

$$F_j = \bigcup\{F(i, t; j) \mid i < j,\ t \in A_i\} .$$

Then it follows from (α) that for $x \in A_i$ and $y \in I_n$ certainly the following holds:

(β) $y \in U_i$ if $y_j \in A_j - F(i, x, j)$ for $j < i$, $y_i = x$, $y_j \in A_j - F_j$ for $j > i$.

Observing that $|F_j| \leqslant \aleph_{j-1} < \aleph_j = |A_j|$ for $j \geqslant 1$, we can construct a point $z \in I_n$ as follows. First choose $z_n \in A_n - F_n$. Then determine $z_{n-1}, z_{n-2}, ..., z_0$ successively in such a way that

$$z_j \in A_j - [F_j \cup \bigcup\{F(k, z_k; j) \mid n \geqslant k > j\}] .$$

Then $z \in U_i$ for each i, by (β), since $z_j \in A_j - F(i, z_i; j)$ for $j < i$, and $z_j \in A_j - F_j$ even for each j. It follows that $U_0 \cap ... \cap U_n \neq \emptyset$.

The proofs that P_n has property D_{2n+1}^{**} are almost the same for $n = 1$ and $n > 1$, so we first consider the case $n = 1$ in detail in 5.3, and then give a hint how to deal with the case $n > 1$ in 5.4.

5.3. Proof that P_1 has property D_3^{**}. Recall that B_1, the "boundary" of P_1, consists of all points $y \in P_1$ with $y_0 = p_0$ or $y_1 = p_1$, and that B_1 is closed. Intuitively it is clear that all we have to do is to construct a linear extender $\Phi : C^*(B_1) \to C^*(P_1)$ such that $\|\Phi(f)\| < 3\|f\|$ for $f \in C^*(B_1)$, provided $\|f\| \neq 0$. We first construct Φ in 5.3.1 and verify that Φ has all properties required in 5.3.2 and 5.3.3, and then confirm our intuition in 5.3.4.

5.3.1. *Construction of* Φ. Consider A_0 to be the set of positive integers, and define for each $f \in C^*(B_1)$ a function $\Phi(f) : P_1 \to \mathbf{R}$ by

$$(1) \qquad \Phi(f)(x) = \begin{cases} f(x) & \text{if } x \in B_1, \\ f(p_0,p_1) + (f(x_0,p_1) - f(p_0,p_1)) \\ \qquad + (1 - 1/x_0)(f(p_0,x_1) - f(p_0,p_1)) \\ \qquad \text{if } x \in I_1 = P_1 - B_1. \end{cases}$$

[The factor $(1 - 1/x_0)$ ensures that $\|\Phi(f)\| < 3\|f\|$ provided $\|f\| \neq 0$; see 5.3.3.] Obviously Φ is linear. Observe that for $x \in I_1$,

$$(2) \qquad \Phi(f)(x) = f(x_0,p_1) + (1 - 1/x_0) \cdot (f(p_0,x_1) - f(p_0,p_1)),$$

$$(3) \qquad \Phi(f)(x) = f(p_0,x_1) + (f(x_0,p_1) - f(p_0,p_1)) + (1/x_0)(f(p_0,p_1) - f(p_0,x_1)).$$

5.3.2. *Proof that* $\Phi(f)$ *is continuous for* $f \in C^*(B_1)$. Fix $f \in C^*(B_1)$ and let $\epsilon > 0$ be arbitrary. Since f is continuous at $q = \langle p_0, p_1 \rangle$, there are finite subsets $F_0 \subset A_0$ and $F_1 \subset A_1$ such that $|f(x) - f(p_0,p_1)| < \epsilon/2$ for $x \in U \cap B_1$, where $U = (C_0 - F_0) \times (C_1 - F_1)$. Then it follows from (1) that

$$|\Phi(f)(z) - \Phi(f)(p_0,p_1)| < (2 - 1/z_0) \cdot \tfrac{1}{2}\epsilon < \epsilon$$

if $z \in U - B_1 = U \cap I_1$; from (2) that

$$|\Phi(f)(z) - \Phi(f)(s,p_1)| < (1 - 1/z_0) \cdot \tfrac{1}{2}\epsilon < \epsilon$$

if $z \in (\{s\} \times (C_1 - F_1)) - B_1$ and $s \in A_0$; and from (3) that

$$|\Phi(f)(z) - \Phi(f)(p_0,t)| < 2 \cdot \tfrac{1}{2}\epsilon$$

if $z \in ((C_0 - F_0) \times \{t\}) - B_1$, $t \in A_1$ and $z_0 > 4\|f\|/\epsilon$.

Therefore $\Phi(f)$ is continuous at each point of B_1. It follows that $\Phi(f)$ is continuous, since the points of $I_1 = P_1 - B_1$ are isolated.

5.3.3. *Proof that* $\|\Phi(f)\| < 3\|f\|$ *if* $\|f\| \neq 0$. Let f be an arbitrary element of $C^*(B_1)$ with $\|f\| \neq 0$. There is an $n_0 \in A_0$, $n_0 \geq 3$, such that $|f(s, p_1) - f(p_0, p_1)| < \frac{1}{2}\|f\|$ for $s > n_0$. Then for any $z \in I_1$ satisfying $z_0 > n_0$ we have by (3) that

$$|\Phi(f)(z)| \leq (1 + \tfrac{1}{2} + 2/(n_0 + 1))\|f\| \leq 2\|f\|.$$

On the other hand, for any $z \in I_1$ satisfying $z_0 \leq n_0$, we have by (2) that

$$|\Phi(f)(z)| \leq (3 - 2/n_0)\|f\|.$$

Consequently $\|\Phi(f)\| \leq (3 - 2/n_0)\|f\|$.

5.3.4. *The general case.* Now let F be an arbitrary nonempty closed subspace of P_1.

Case 1: $F \cap B_1 = \emptyset$. Then F is both open and closed in P_1, and so there is a retraction $r : P_1 \to F$. Then an extender $\Psi : C^*(F) \to C^*(P_1)$ can be defined by $\Psi(f) = f \circ r$. Clearly Ψ is linear and $\|\Psi\| = 1$.

Case 2: $F \cap B_1 \neq \emptyset$. We first construct a retraction $r : B_1 \to F \cap B_1$. If $\langle p_0, p_1\rangle \notin F$, then $F \cap B_1$ is both open and closed in B_1 and hence the desired retraction exists. If $\langle p_0, p_1\rangle \in F$ we can define the desired retraction r by

$$r(x) = \begin{cases} x & \text{if } z \in F \cap B_1, \\ \langle p_0, p_1\rangle & \text{otherwise}. \end{cases}$$

Let $\Phi : C^*(B_1) \to C^*(P_1)$ be the extender defined in 5.3.1. For each $f \in C^*(F)$, $(f \restriction F \cap B_1) \circ r$ belongs to $C^*(B_1)$, hence we can define a function $\Psi(f) : P_1 \to R$ by

$$\Psi(f)(x) = \begin{cases} f(x) & \text{if } x \in F, \\ \Phi((f \restriction F \cap B_1) \circ r)(x) & \text{if } x \notin F. \end{cases}$$

Clearly, $\Psi(f)$ is continuous on the open set $P_1 - F$. Also, $\Psi(f)$ is continuous at each point of $F - B_1$ (since such a point is isolated in P_1).

Next, let $x \in F \cap B_1$. Then both f and $\Phi((f \restriction F \cap B_1) \circ r)$ are continuous at x, while $f(x) = \Phi((f \restriction F \cap B_1) \circ r)(x)$; hence f is continuous at x. It follows that $\Psi(f)$ is continuous. Finally, from the definition of Ψ and from 5.3.3 we conclude that $\|\Psi(f)\| < 3\|f\|$ if $\|f\| \neq 0$; and evidently Ψ is linear.

5.4. The case $n > 1$. We now indicate how to deal with the case $n > 1$. Write $f[i, x]$ for $f(e(i, x))$, and replace 5.3.1(1) by

$$\Phi(f)(x) = \begin{cases} f(x) & \text{if } x \in B_n\,, \\ f(q)+(f[0,x_0]-f(q))+(1-1/x_0)\sum_{i=1}^{n}(f[i,x_i]-f(q)) \\ \qquad \text{if } x \in P_n - B_n\,. \end{cases}$$

Then the proof that P_1 has property D_3^{**} can easily be adapted to show that P_n has property D_{2n+1}^{**}, also for $n > 1$.

5.5. Remark. The idea of Lemma 3.2 stems from an investigation of example P_1. It is not difficult to find the extender $\Phi: C^*(B_1) \to C^*(P_1)$, defined by

$$\Phi(f)(x) = \begin{cases} f(x) & \text{if } x \in B_1\,, \\ f(p_0,p_1)+(f(x_0,p_1)-f(p_0,p_1))+(f(x_1,p_0)-f(p_0,p_1)). \\ \text{if } x \in P_1 - B_1\,. \end{cases}$$

For $x \in I_1 = P_1 - B_1$, define $f_x \in C^*(B_1)$ by

$$f_x(y) = \begin{cases} 1 & \text{if } y_0 = x_0 \text{ or } y_1 = x_1\,, \\ -1 & \text{otherwise}\,. \end{cases}$$

Then $\|f_x\| = 1$ but $\|\Phi(f_x)\| = 3$. By a cardinality argument it can be shown that for every linear extender $\Phi: C^*(B_1) \to C^*(P_1)$ and for every $c < 3$ there is an $x \in I_1$ such that $\Phi(f_x)(x) > c$. The observation that this involves functions which change sign led to the discovery of Lemma 3.2.

6. Questions

The spaces S and X in 2.3, and the subspace $P_1 - \{\langle p_0, p_1\rangle\}$ of P_1 are examples of K_2-spaces which do not have any property D_c^*, since they are not normal.

(1) *Must a K_1-space have property D_1^*?*

We believe that K_1-spaces need not have any property D_c^*.

In all known examples, subspaces of a space having property D_1^* also have property D_1^*.

(2) *Is property D_1^* heriditary? Is property D_c^* hereditary in hereditarily normal spaces?*

Our example P_1 shows that property D_3^* is not hereditary. In [7] we will use an example of Heath and Lutzer [15] to show that some extension properties, strong enough to imply that a space is a K_1-space, are not hereditary; for example the conclusion of the Dugundji Extension Theorem. Note that a positive answer to the first question settles the first part of the second question positively. However, we believe that property D_1^* is not hereditary.

The spaces P_n are paracompact; there are spaces with property D_1^* which are not paracompact, e.g. the countable ordinals, [15], see also [7]. So all known examples of spaces with some property D_c^* are collectionwise normal.

(3) *Must a space which has any property D_c^* be collectionwise normal? Or is Corollary 3.3 best possible?*

(4) *Does there exist for each real number $\lambda > 1$ a space which has property D_λ^* but not property D_c^* for any $c < \lambda$?*

Appendix

In this appendix we indicate how to construct an $\aleph_0$-space M_∞ which does not have any property D_c^*. We will not repeat the definition of an $\aleph_0$-space, because we only need the following facts:

(1) Any separable metrizable space is an $\aleph_0$-space [19, (A)].

(2) If the regular space X is covered by a countable collection $\mathcal{A}$ of closed $\aleph_0$-spaces, and if each compact subset of X is contained in

the union of some finite subcollection of $\mathscr{A}$, then X is an $\aleph_0$-space [19, Proposition 7.7].

As in Section 4, we will construct for each $n \in \mathrm{N}$ an $\aleph_0$-space M_n which is not a K_n-space. Then M_∞ will be the topological sum of the M_n's. The basic idea for our construction is to combine the construction of the spaces H_n with Michael's method of turning a butterfly space into an $\aleph_0$-space [19, Example 12.7].

A, B and C are defined as in Section 4. Q is any subspace of $\beta\mathrm{N}$ which contains N and exactly one point, q, of $\beta\mathrm{N} - \mathrm{N}$. Observe that:

(3) Each infinite subset of N contains an infinite subset which is both open and closed in Q.

Construction of M_1. M_1 is the subspace

$$(A \times \{q\}) \cup ((H_1 - A) \times \mathrm{N})$$

of $H_1 \times Q$, cf. [19, Example 12.7]. Then $\{A \times \{q\}\} \cup \{(H_1 - A) \times \{i\} \mid i \in \mathrm{N}\}$ is a countable closed cover of M_1, each member of which is an $\aleph_0$-space because of (1). Clearly (3) implies that no compact subset of M_1 intersects $(H_1 - A) \times \{i\}$ for infinitely many i. Hence M_1 is an $\aleph_0$-space (2). The proof that M_1 is not a K_1-space is entirely analogous to the proof that H_1 is not a K_1-space: Let $G_1 = M_1 \cap (F_1 \times Q)$, let $\kappa : \tau G_1 \to \tau M_1$ be any function such that $G_1 \cap \kappa(V) = V$ for $V \in \tau G_1$, and consider

$$L_{k,i} = \{x \in A \mid U_1(x, k) \times \{i\} \subset \kappa(G_1 \cap (U_1(x, 1) \times Q))\}$$

instead of L_k.

Construction of M_{n+1} for $n \in \mathrm{N}$. Assume that we have constructed an $\aleph_0$-space M_n which is not a K_n-space, in which the (necessarily countable) set of isolated points is dense. We first construct an additional space S_n. Although M_n is not first countable (for otherwise it would be metrizable [17a, (B)]), it will be clear how to adapt the construction of H_{n+1} so as to get a space S_n the underlying set of which is $A \cup (B \times M_n) \cup C$. Denote $B \times M_n$ by B_n'. Then the underlying set of M_{n+1} is

$$(A \times \{q\} \times \{q\}) \cup (B_n' \times \mathrm{N} \times \{q\}) \cup (C \times \mathrm{N} \times \mathrm{N})$$

It is possible to consider M_{n+1} as a subspace of $S_{n+1} \times Q \times Q$, but for the proof that M_{n+1} is not a K_{n+1}-space it seems more convenient to topologize M_{n+1} as follows: points of $M_{n+1} - (A \times \{q\} \times \{q\})$ have their subspace-of-$S_{n+1} \times Q \times Q$-neighborhoods, but a basic neighborhood of a point $\langle x, q, q\rangle \in A \times \{q\} \times \{q\}$ has the form M_{n+1} $M_{n+1} \cap (U \times V \times \mathbf{N})$, where U and V are neighborhoods of x in S_{n+1} and of q in Q, respectively (instead of $M_{n+1} \cap (U \times V \times V)$). The proof that M_{n+1} is not a K_{n+1}-space is left to be reader.

The collection

$$\mathcal{A} = \{C \times \{i\} \times \{j\} \mid i, j \in \mathbf{N}\} \cup \{B'_n \times \{i\} \times \{q\} \mid i \in \mathbf{N}\} \\ \cup \{A \times \{q\} \times \{q\}\}$$

is a countable closed cover of M_{n+1}, each member of which is an $\aleph_0$-space. [Observe that B'_n is a topological sum of countably many $\aleph_0$-spaces, hence is an $\aleph_0$-space by (2).] Let F be a compact subset of M_{n+1}. Because of (3), F intersects $S_n \times \{i\} \times Q$ for at most finitely many $i \in \mathbf{N}$. For each such i this intersection is compact, hence F intersects $C \times \{i\} \times \{j\}$ for at most finitely many $j \in \mathbf{N}$, again because of (3). It now follows from (2) that M_{n+1} is an $\aleph_0$-space.

Added in proof. Let $A = ([0,1] \times \{0,1\}) - \{\langle 0,0\rangle, \langle 1,1\rangle\}$ have the topology induced by the lexicographic order, and let S be a convergent sequence. Then $A \times S$ is a compact Hausdorff space which is hereditarily Lindelöf. However, the proof of 4.2 can easily be adapted to show that $A \times S$ is not a K_1-space, hence $A \times S$ does not have property D_c^* for any $c < 3$.

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